Name ____ Date ____________ Class _____________

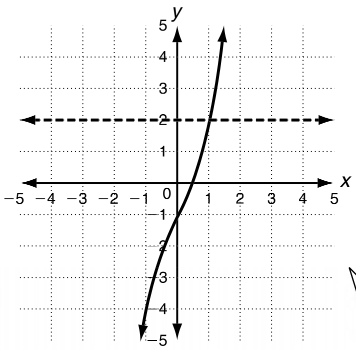
LESSON 9-5

Reteach

Functions and Their Inverses

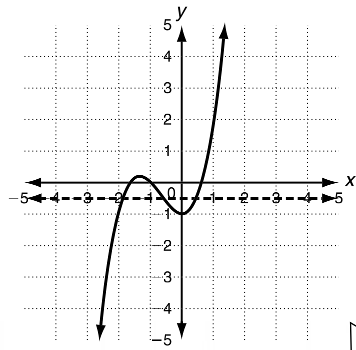
Not all relations are functions and not all relations have inverse functions. To decide whether the inverse of a relation is a function, use the **horizontal-line test.**

If any horizontal line passes through more than one point on the graph of a relation, the inverse of the relation is not a function.



Can you draw a horizontal line that passes through more than one point on the graph?

No. So the inverse of the relation is a function.

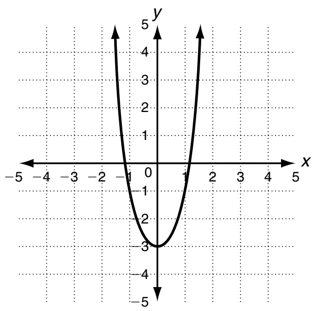


Can you draw a horizontal line that passes through more than one point on the graph?

Yes. So the inverse of the relation is **NOT a** function.

Use the horizontal-line test to determine whether the inverse of each relation is a function.

1.



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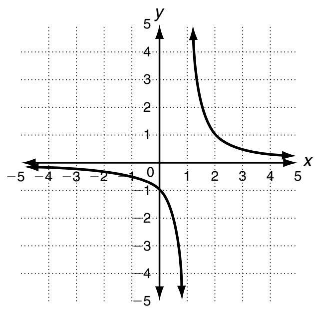
2.



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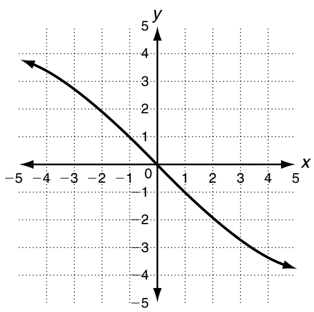
3.



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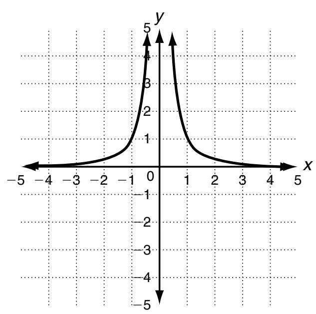
4.



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Name ______________________________ Date __________ Class __________

LESSON 9-5

Reteach

Functions and Their Inverses (continued)

The inverse of a function, $f(x)$, is denoted $f^{-1}(x)$.
To find the inverse of a function $f(x)$, write $y = f(x)$. Then switch x and y and solve the equation for y.

Find the inverse of $f(x) = x^2 - 1$.

Step 1 Write $y = f(x)$.

$$y = x^2 - 1$$

Step 2 Switch x and y.

$$x = y^2 - 1$$

Step 3 Solve for y.

$$x = y^2 - 1$$

$$x + 1 = y^2$$ — Isolate the variable, y.

$$\sqrt{x + 1} = \sqrt{y^2}$$ — Take the square root of both sides.

$$\pm\sqrt{x + 1} = y$$ — The domain is $\{x \mid x \geq -1\}$.

Remember, $\sqrt{x + 1}$ can be either positive or negative.

Because y could have 2 different x-values, $\sqrt{x + 1}$ and $-\sqrt{x + 1}$, this inverse is not a function. Check by graphing the original function on a graphing calculator and use the horizontal-line test.

The domain of the inverse is the range of $f(x)$: $\{x \mid x \geq -1\}$.

The range of the inverse is the domain of $f(x)$: all real numbers.

Find the inverse of each function and state its domain and range.

5. $f(x) = \sqrt{x} + 3$

$y = \sqrt{x} + 3$

$x = \sqrt{y} + 3$

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6. $f(x) = 2x + 1$

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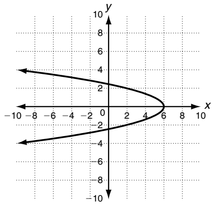
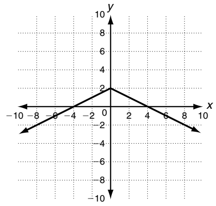
LESSON 9-5
Practice A
Functions and Their Inverses

Use the horizontal line test to determine whether the inverse of each relation is a function.

1. 

Yes

2. 

No

Find the inverse of each function. State whether the inverse is a function.

3. $f(x) = 2x - 6$
   a. Substitute y for $f(x)$. $y = 2x - 6$
   b. Switch x and y. $x = 2y - 6$
   c. Solve for y. This is the inverse. $y = \frac{x+6}{2}$
   d. Graph the original function using a graphing calculator. Is the inverse a function? yes

4. $g(x) = x + 9$ — $g^{-1}(x) = x - 9$; yes

5. $h(x) = \frac{x}{2} - 1$ — $h^{-1}(x) = 2x + 2$; yes

6. $p(x) = x^2 + 1$ — $p^{-1}(x) = \pm\sqrt{x-1}$; no

7. $b(x) = \sqrt{x+7}$ — $b^{-1}(x) = x^2 - 7$; yes, for $x \geq -7$

Solve.

8. The total cost of a jacket, including 8% tax, can be found by using the function $T(x) = 1.08x$.
   a. Find the inverse of $T(x)$. $T^{-1}(x) = \frac{x}{1.08}$
   b. What does the inverse represent? The price of the jacket
   c. Tia wants to return a jacket. She paid a total of \$102.60 for it. What was the price of the jacket? \$95

LESSON 9-5
Practice B
Functions and Their Inverses

Find the inverse of each function. Determine whether the inverse is a function and state its domain and range.

1. $k(x) = 10x + 5$
   $k^{-1}(x) = \frac{x-5}{10}$; function
   domain: $(-\infty, +\infty)$
   range: $(-\infty, +\infty)$

2. $d(x) = 6 - 2x$
   $d^{-1}(x) = -\frac{x}{2} + 3$; function
   domain: $(-\infty, +\infty)$
   range: $(-\infty, +\infty)$

3. $f(x) = (x - 5)^2$
   $y = 5 \pm \sqrt{x}$; not a function
   domain: $(-\infty, +\infty)$
   range: $[0, +\infty)$

4. $g(x) = \frac{4-x}{2}$
   $g^{-1}(x) = -2x + 4$; function
   domain: $(-\infty, +\infty)$
   range: $(-\infty, +\infty)$

5. $h(x) = \sqrt{x^2 - 9}$
   $h^{-1}(x) = \pm\sqrt{x^2 + 9}$; not a function
   domain: $[0, +\infty)$
   range: $(-\infty, -3]$ and $[3, +\infty)$

6. $b(x) = 2\log x$
   $b^{-1}(x) = \log^{-1}\frac{x}{2}$ or $b^{-1}(x) =$ $10^{\frac{x}{2}}$; function; domain: $(-\infty, +\infty)$
   range: $[0, +\infty)$

Determine by composition whether each pair of functions are inverses.

7. $q(x) = \sqrt{x} - 4$ and $r(x) = x^2 + 4$ for $x \geq 0$ — No

8. $s(x) = \frac{2}{x-2}$ and $t(x) = \frac{x+2}{-2}$ — No

9. $u(x) = \frac{x^2}{4} - 1$ for $x \geq -1$ and $v(x) = \pm 2\sqrt{x+1}$ — Yes

10. $A(x) = \log(x-1)^4$ and $B(x) = 1 + \log^{-1}\left(\frac{x}{4}\right)$ — Yes

Solve.

11. So far, Rhonda has saved \$3000 for her college expenses. She plans to save \$30 each month. Her college fund can be represented by the function $f(x) = 30x + 3000$.
    a. Find the inverse of $f(x)$. $f^{-1}(x) = \frac{1}{30}x - 100$
    b. What does the inverse represent? Number of months she has saved
    c. When will the fund reach \$3990? 33 months
    d. How long will it take her to reach her goal of \$4800? 5 years

LESSON 9-5
Practice C
Functions and Their Inverses

Find the inverse of each function. Determine whether the inverse is a function and state its domain and range.

1. $A(x) = \frac{9-2x}{5}$
   $A^{-1}(x) = -2.5x + 4.5$; function
   domain: $(-\infty, +\infty)$
   range: $(-\infty, +\infty)$

2. $B(x) = \frac{3+x}{x}$
   $B^{-1}(x) = \frac{3}{x-1}$; function
   domain: $(-\infty, 1)$ and $(1, +\infty)$
   range: $(-\infty, 0)$ and $(0, +\infty)$

3. $C(x) = 25 - x^2$
   $C^{-1}(x) = \pm\sqrt{25-x}$
   not a function; domain: $(-\infty, 25]$
   range: $(-\infty, +\infty)$

4. $D(x) = 2 - \log x^3$
   $D^{-1}(x) = \log^{-1}\left(\frac{-x+2}{3}\right)$; not a function; domain: $(-\infty, +\infty)$
   range: $(-\infty, 0)$ and $(0, +\infty)$

5. $E(x) = \frac{x}{x+2}$
   $E^{-1}(x) = \frac{2x}{1-x}$; not a function
   domain: $(-\infty, -2)$ and $(-2, +\infty)$
   range: $(-\infty, 1)$ and $(1, +\infty)$

6. $F(x) = 4 + \sqrt{2x-1}$
   $F^{-1}(x) = 0.5x^2 - 4x + 8.5$
   function; domain: $[4, +\infty)$
   range: $[0.5, +\infty)$

Determine by composition whether each pair of functions are inverses.

7. $p(x) = \sqrt{5 - x^2}$ for $|x| \leq 5$ and $q(x) = \sqrt{5 - x^2}$ for $|x| \leq 5$ — no

8. $s(x) = \frac{-2x}{x-2}$ and $t(x) = \frac{2x}{2-x}$ — yes

9. $u(x) = \frac{1}{(x-3)^2}$, for $x > 3$ and $v(x) = 3 + \frac{\sqrt{x}}{x}$, for $x > 0$ — yes

10. $b(x) = \log(x-1)^4$ and $d(x) = 1 + 10^{\frac{x}{4}}$ for $x \geq 1$ — yes

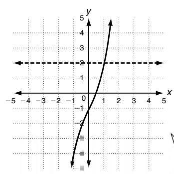
Solve.

11. The area of a regular octagon can be found by using the formula $A(s) = 2s^2(\sqrt{2} + 1)$, where s is the length of each side.
    a. Find the inverse of $A(s)$. $s = \pm\sqrt{\frac{A}{2(\sqrt{2}+1)}}$
    b. What does the inverse represent? Side length
    c. What is the side length of a regular octagon whose area is $(9.68\sqrt{2} + 9.68)$ m? 2.2 meters

LESSON 9-5
Reteach
Functions and Their Inverses

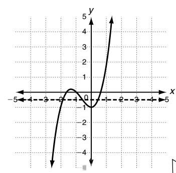
Not all relations are functions and not all relations have inverse functions. To decide whether the inverse of a relation is a function, use the **horizontal-line test.**

If any horizontal line passes through more than one point on the graph of a relation, the inverse of the relation is not a function.



Can you draw a horizontal line that passes through more than one point on the graph?

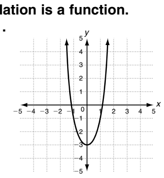
No. So the inverse of the relation is a function.



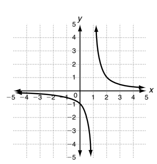
Can you draw a horizontal line that passes through more than one point on the graph?

Yes. So the inverse of the relation is **NOT a** function.

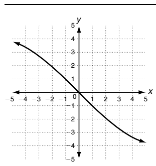
Use the horizontal-line test to determine whether the inverse of each relation is a function.

1. 

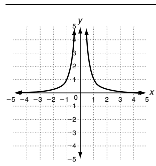
Not a function

2. 

Function

3. 

Function

4. 

Not a function

LESSON **9-5**

Reteach

Functions and Their Inverses (continued)

The inverse of a function, $f(x)$, is denoted $f^{-1}(x)$.
To find the inverse of a function $f(x)$, write $y = f(x)$. Then switch x and y and solve the equation for y.

Find the inverse of $f(x) = x^2 - 1$.

Step 1 Write $y = f(x)$.

$y = x^2 - 1$

Step 2 Switch x and y.

$x = y^2 - 1$

Step 3 Solve for y.

$x = y^2 - 1$

Isolate the variable, y.

$x + 1 = y^2$

Take the square root of both sides.

$\sqrt{x+1} = \sqrt{y^2}$

The domain is $\{x \mid x \geq -1\}$.

$\pm\sqrt{x+1} = y$

Remember, $\sqrt{x+1}$ can be either positive or negative.

Because y could have 2 different x-values, $\sqrt{x+1}$ and $-\sqrt{x+1}$, this inverse is not a function. Check by graphing the original function on a graphing calculator and use the horizontal-line test.

The domain of the inverse is the range of $f(x)$: $\{x \mid x \geq -1\}$.

The range of the inverse is the domain of $f(x)$: all real numbers.

Find the inverse of each function and state its domain and range.

5. $f(x) = \sqrt{x} + 3$

$y = \sqrt{x} + 3$

$x = \sqrt{y} + 3$

$(x - 3) = \sqrt{y}$

$f^{-1}(x) = (x - 3)^2$

domain: all real numbers x such that $x \geq 3$; range: nonnegative real numbers

6. $f(x) = 2x + 1$

$y = 2x + 1$

$x = 2y + 1$

$x - 1 = 2y$

$f^{-1}(x) = \frac{x - 1}{2}$

domain: all real numbers; range: all real numbers

LESSON **9-5**

Challenge

One-to-One Functions

When both a relation and its inverse are functions, the relation is called a one-to-one function. The horizontal line test can be used to determine if the inverse of a relation is a function. But sometimes it is difficult to graph the relation in order to use the horizontal line test. Therefore, an alternate definition is useful.

A function is a one-to-one function if whenever a and b are in the domain of the function f and $a \neq b$, then $f(a) \neq f(b)$.

This definition may be used to determine whether a function is one-to-one or not. Consider the two functions $f(x) = 3x + 8$ and $g(x) = x^2 - 4$. The domain for both of the functions is the set of all real numbers.

For the function f, if $a \neq b$, then certainly $3a \neq 3b$ and likewise $3a + 8 \neq 3b + 8$; therefore, $f(a) \neq f(b)$ and f is a one-to-one function.

For the function g, does $a \neq b$ imply that $a^2 \neq b^2$? No since $-1 \neq 1$ but $g(-1) = g(1)$. Therefore, the function g is not a one-to-one function.

Use the horizontal line test and the definition of a one-to-one function given above to determine whether the following functions are one-to-one on the domain specified. Most of these may be graphed on a calculator, but remember to consider the domain specified and not the domain the calculator uses.

1. $f(x) = \frac{1}{2}x + 9$, domain is the set of real numbers.

Linear functions with a nonzero slope are always one-to-one.

2. $f(x) = \frac{1}{2}x + 9$, domain is the set of positive integers.

Linear functions with a nonzero slope are always one-to-one.

3. $f(x) = (x - \sqrt{2})^2$, domain is the set of real numbers.

Not one-to-one; fails the horizontal line test; also try $\sqrt{2} + 1$ and $\sqrt{2} - 1$ for a and b.

4. $f(x) = (x - \sqrt{2})^2$, domain is the set of integers. One-to-one

5. $f(x) = x^3 + x^2 - 2x - 1$, domain is the set of real numbers.

Fails horizontal line test, not one-to-one

6. $f(x) = x^3 + x^2 - 2x - 1$, domain is the set of integers. Not one-to-one, $f(-2) = f(0)$

7. $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$, domain is the set of integers. Not one-to-one; $f(0) = f(1) = 0$

8. $f(x) = \lceil 2x \rceil$, domain is the set of integers. One-to-one

9. $f(x, y) = \max(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2}$, where (x, y) are ordered pairs of positive integers.

Not one-to-one since $(3, 5)$ is not equal to $(4, 5)$ but $\max(3, 5) = 5 = \max(4, 5)$

LESSON **9-5**

Problem Solving

Functions and Their Inverses

A juice drink manufacturer is designing an advertisement for a national sports event on its cans. The lateral surface area of the cans is given by the function $L(h) = 2.5\pi h$, where h is the height of the can. The total surface area of the can is given by the function $T(h) = 2.5\pi(h + 1.25)$.

1. The graphic designer needs to know how the height of the can varies as a function of the lateral surface area.

a. Find the inverse, $h(L)$, of the function $L(h)$.

$L(h) = 2.5\pi h$; $h(L) = \frac{L}{2.5\pi}$

b. Explain the meaning of the inverse function.

$h(L)$ gives the height of a can for a given lateral surface area.

c. If the lateral surface area of one can is 35.34 in^2, what is the height of this can? 4.5 in.

Choose the letter for the best answer.

2. The manufacturer produces cans in different sizes. The height of one can is 5.5 in. The designer is planning to use only half the lateral surface area of this can. What is this area?

A 8.8 in^2

B 11.0 in^2

(C) 21.6 in^2

D 43.2 in^2

3. The designer is studying the possibility of using the total surface area of each can. Which function gives height, $h(T)$, as a function of total surface area, T?

(F) $h(T) = \frac{T}{2.5\pi} - 1.25$

G $h(T) = \frac{T}{2.5\pi} + 1.25$

H $h(T) = (2.5\pi)(T - 1.25)$

J $h(T) = (2.5\pi)(T + 1.25)$

4. The total surface area of one size of can is 45.16 in^2. What is the height of this can?

A 3.5 in.

(B) 4.5 in.

C 5.5 in.

D 6.5 in.

5. The designer updates an old advertisement that covers the lateral surface area, L, of a can to create a new advertisement that covers the total surface area, T, of the can. Which function gives this area?

F $T = (2.5\pi)(L + 1.25)$

G $T = (2.5\pi)(L - 1.25)$

H $T = L - 2.5\pi(1.25)$

(J) $T = L + 2.5\pi(1.25)$

LESSON **9-5**

Reading Strategies

Analyze Information

Looking at the graph of a function, you can determine whether its inverse is a function. The vertical-line test is used to check whether the graph is a function. A **horizontal-line test** is used to check whether the inverse of the function is also a function. This test allows you to find whether the inverse is a function without drawing the graph of the inverse function.

Vertical-line Test	Horizontal-line Test
If any vertical line intersects the graph at only one point, then the graph represents a function.	If any horizontal line intersects the graph at only one point, then the inverse relation is a function.

Solve.

1. The function $f(x) = 3$ is a linear function. Explain whether the inverse of this graph is a function.

The inverse is not a function. The graph of f is a horizontal line. So the horizontal line $y = 3$ intersects the graph of f at infinitely many points.

2. All linear functions $f(x) = mx + b$, where $m \neq 0$, have an inverse relation that is a function. Use the horizontal-line test to explain why.

If the slope is not 0, then the graph is a straight line inclined at an angle. All horizontal lines will intersect the graph at only one point, so these graphs will all have inverse relations that are functions.

3. a. The points $(-1, 4)$ and $(5, 4)$ are on the graph of a function g. What happens if you draw a horizontal line $y = 4$ through the graph of the function?

The line $y = 4$ will intersect the function at least at two points.

b. Is the inverse of the function g also a function? Explain why.

No; the function fails the horizontal-line test.

4. Suppose the graph of the inverse of a function is given. Would you use the horizontal-line test or the vertical-line test to determine whether the graph is a function? Explain.

The vertical-line test should be used to test whether the graph represents a function.