### Reteach

**LESSON 4-4**

**Determinants and Cramer's Rule**

A square matrix has the same number of rows as columns. The determinant of a square matrix is shown by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

To find the determinant of a $2 \times 2$ matrix, find the product of each diagonal, beginning at the upper left corner. Then subtract.

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

$$\det \begin{vmatrix} 2 & 3 \\ 5 & 9 \end{vmatrix} = 2(9) - 5(3) = 18 - 15 = 3$$

**Find the determinant of each matrix.**

1. $\det \begin{vmatrix} -1 & 2 \\ -5 & 4 \end{vmatrix} = -1(4) - (-5)(2) = 

2. $\det \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} = 3(-4) - (-1)(-4) = 

3. $\det \begin{vmatrix} -3 & 4 \\ -1 & 6 \end{vmatrix} = -3(6) - (-1)(-4) = 

4. $\det \begin{vmatrix} -2.4 & 0.5 \\ 1.2 & 2 \end{vmatrix} = 

5. $\det \begin{vmatrix} 1 & 9 \\ 3 & 12 \end{vmatrix} = 

6. $\det \begin{vmatrix} 8 & 2 \\ -15 & 3 \end{vmatrix} = 

Vertical brackets indicate a determinant.
Use Cramer’s rule to solve a system of linear equations. \[
\begin{align*}
x + y &= 2 \\
y + 7 &= 2x
\end{align*}
\]

**Step 1** Write the equations in standard form, \(ax + by = c\).

\[
\begin{align*}
x + y &= 2 \\
2x - y &= 7
\end{align*}
\]

**Step 2** Write the coefficient matrix of the system of equations. Then find the determinant of the coefficient matrix.

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
2 & -1
\end{bmatrix}
\]

\[
D = \frac{c_1}{a_1} = \frac{1}{1} = 1
\]

**Step 3** Solve for \(x\) and \(y\) using Cramer’s rule. Remember to divide by the determinant.

\[
x = \frac{c_1}{a_1} = \frac{2}{1} = 2
\]

\[
y = \frac{c_2}{a_2} = \frac{-3}{-1} = 3
\]

The solution is \((3, -1)\).

Use Cramer’s rule to solve each system of equations.

7. \[
\begin{align*}
2x + y &= -1 \\
4x + y &= -5
\end{align*}
\]

\[
D = \begin{vmatrix}
2 & 1 \\
4 & 1
\end{vmatrix} = 6
\]

\[
x = \frac{1}{6} = \frac{-1}{6}
\]

\[
y = \frac{-5}{6}
\]

8. \[
\begin{align*}
x - y &= 1 \\
3x - 2y &= 4
\end{align*}
\]

\[
D = \begin{vmatrix}
1 & -1 \\
3 & -2
\end{vmatrix} = 7
\]

\[
x = \frac{1}{7} = \frac{1}{7}
\]

\[
y = \frac{-2}{7}
\]

9. \[
\begin{align*}
y - x &= 3 \\
2x - 2 &= y
\end{align*}
\]

\[
D = \begin{vmatrix}
1 & -1 \\
2 & 1
\end{vmatrix} = 3
\]

\[
x = \frac{3}{3} = 1
\]

\[
y = \frac{-2}{3}
\]

10. \[
\begin{align*}
3y &= 4x + 7 \\
9 - 6x &= 2y
\end{align*}
\]

\[
D = \begin{vmatrix}
4 & 7 \\
-6 & 2
\end{vmatrix} = -22
\]

\[
x = \frac{-22}{-22} = 1
\]

\[
y = \frac{9}{-22}
\]
**Practice A**

**Determinants and Cramer’s Rule**

Find the determinant of each matrix.

1. \[
\begin{vmatrix}
5 & 2 \\
1 & -3
\end{vmatrix}
\]

2. \[
\begin{vmatrix}
6 & -1 \\
2 & 5
\end{vmatrix}
\]

3. \[
\begin{vmatrix}
3 & 0 \\
2 & 3
\end{vmatrix}
\]

Use Cramer’s rule to solve each system of equations.

4. \[
\begin{align*}
x + y &= 5 \\
x - 2y &= 0
\end{align*}
\]

5. \[
\begin{align*}
x + y &= 6 \\
x - 2y &= -3
\end{align*}
\]

6. \[
\begin{align*}
x + y &= 5 \\
x - 2y &= 3
\end{align*}
\]

**Practice B**

**Determinants and Cramer’s Rule**

Find the determinant of each matrix.

1. \[
\begin{vmatrix}
1 & 2 \\
4 & -1
\end{vmatrix}
\]

2. \[
\begin{vmatrix}
6 & 3 \\
9 & -5
\end{vmatrix}
\]

3. \[
\begin{vmatrix}
2 & 8 \\
6 & 3
\end{vmatrix}
\]

Use Cramer’s rule to solve each system of equations.

4. \[
\begin{align*}
x + y &= 1 \\
x - 2y &= 0
\end{align*}
\]

5. \[
\begin{align*}
x + y &= 5 \\
x - 2y &= 3
\end{align*}
\]

6. \[
\begin{align*}
x + y &= 4 \\
x - 2y &= 1
\end{align*}
\]

**Practice C**

**Determinants and Cramer’s Rule**

Find the determinant of each matrix.

1. \[
\begin{vmatrix}
12 & 5 \\
14 & -3
\end{vmatrix}
\]

2. \[
\begin{vmatrix}
6 & -1 \\
2 & 5
\end{vmatrix}
\]

3. \[
\begin{vmatrix}
4 & 11 \\
3 & 1
\end{vmatrix}
\]

Use Cramer’s rule to solve each system of equations.

4. \[
\begin{align*}
x + y &= 3 \\
x - 2y &= 1
\end{align*}
\]

5. \[
\begin{align*}
x + y &= 4 \\
x - 2y &= 1
\end{align*}
\]

6. \[
\begin{align*}
x + y &= 1 \\
x - 2y &= 2
\end{align*}
\]

7. \[
\begin{align*}
x + y &= 2 \\
x - 2y &= 0
\end{align*}
\]

8. \[
\begin{align*}
x + y &= 5 \\
x - 2y &= 3
\end{align*}
\]

**Retain**

**Determinants and Cramer’s Rule**

A square matrix has the same number of rows as columns. The determinant of a square matrix is shown by \( \det \).

To find the determinant of a \( 2 \times 2 \) matrix, find the product of each diagonal, beginning at the upper left corner. Then subtract.

\[
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc
\]

**Find the determinant of each matrix.**

1. \[
\begin{pmatrix}
-1 & 2 \\
-4 & 1
\end{pmatrix}
\]

2. \[
\begin{pmatrix}
3 & -1 \\
4 & 2
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
2 & 3 \\
-1 & 2
\end{pmatrix}
\]

4. \[
\begin{pmatrix}
-2 & 4 \\
4 & -3
\end{pmatrix}
\]

5. \[
\begin{pmatrix}
9 & 3 \\
-6 & 2
\end{pmatrix}
\]

6. \[
\begin{pmatrix}
8 & 2 \\
3 & 1
\end{pmatrix}
\]

Solve.

13. Travis invested $20,000 in two simple interest accounts. He invested part at 4.5% interest and the rest at 3.5% interest. He earned $785 in total interest per year.

a. Write the problem as a system of equations.

b. Find the value of the determinant of the coefficient matrix.

c. Use Cramer’s rule to write the solution for the amount Travis invested at 4.5%.

d. How much did Travis invest at 4.5% interest?
Determinants and Cramer’s Rule (continued)

Use Cramer’s rule to solve a system of linear equations:

\[ \begin{align*}
2x + y &= 7 \\
2x - y &= 2
\end{align*} \]

Step 1: Write the equations in standard form, \( ax + by = c \).

\[ \begin{align*}
x + y &= 7 \\
2x - y &= 2
\end{align*} \]

Step 2: Write the coefficient matrix of the system of equations. Then find the determinant of the coefficient matrix.

\[
\begin{vmatrix}
1 & 1 \\
2 & -1
\end{vmatrix} = -3
\]

\[ D = \begin{vmatrix}
1 & 1 \\
2 & -1
\end{vmatrix} = -3
\]

Step 3: Solve for \( x \) and \( y \) using Cramer’s rule. Remember to divide by the determinant.

\[
\begin{align*}
x &= \frac{\begin{vmatrix}
7 & 1 \\
2 & -1
\end{vmatrix}}{-3} = \frac{-5}{-3} = \frac{5}{3} \\
y &= \frac{\begin{vmatrix}
1 & 7 \\
2 & 2
\end{vmatrix}}{-3} = \frac{-8}{-3} = \frac{8}{3}
\end{align*} \]

The solution is \( \left( \frac{5}{3}, \frac{8}{3} \right) \).

Determinants and Variations with Matrix Operations

What happens to the determinant of a matrix as the entries in the matrix are changed? Certain changes affect the value of the determinant and others do not. These operations that do not change the determinant are called invariant operations. The determinant of matrix \( R \) is 36.

1. Calculate the determinant of matrix \( S \).

\[
R = \begin{vmatrix}
1 & 0 & -2 \\
3 & 2 & 5
\end{vmatrix}
\]

\[
S = \begin{vmatrix}
1 & 0 & -2 \\
3 & 2 & 5
\end{vmatrix}
\]

\[ \text{Determinant of } R = 36 \]

\[ \text{Determinant of } S = \text{some value} \]

2. Interchange the first and second rows to get matrix \( T \).

\[
T = \begin{vmatrix}
3 & 2 & 5 \\
1 & 0 & -2
\end{vmatrix}
\]

\[ \text{Determinant of } T = \text{some value} \]

3. Multiply the first row of matrix \( R \) by 3. Now find the new determinant. How does this value compare to the original determinant?

\[
R' = \begin{vmatrix}
3 & 0 & -6 \\
3 & 2 & 5
\end{vmatrix}
\]

\[ \text{Determinant of } R' = \text{some value} \]

Challenge

Challenge:

1. Multiply matrix \( R \) by 3 to get matrix \( T \). Find the determinant of matrix \( T \).

\[ \text{Determinant of } T = \text{some value} \]

2. Multiply the matrix by 4 and find the determinant again. Write a conjecture about how the determinant of a matrix changes when the matrix is multiplied by a constant.

\[ \text{Determinant of } 4R = \text{some value} \]

3. Use matrix \( R \) and add twice the first row to the second row. This becomes the new second row. Write the new matrix \( U \). Find its determinant.

\[ \text{Determinant of } U = \text{some value} \]

4. Try this with another \( 3 \times 3 \) matrix. What conjecture can you make about how this operation affects the determinant?

Possible answer: This is an invariant operation. Adding a multiple of one row to another row does not change the determinant.

Reading Strategies

Analyze Information

Every square matrix has a determinant. The determinant can be positive, negative, or 0. The determinant of a \( 2 \times 2 \) matrix is the difference of the product of the diagonals. Always subtract from the diagonal that starts in the upper left of the matrix.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Determinant</th>
</tr>
</thead>
</table>
| \[
\begin{vmatrix}
1 & 2 \\
3 & 4
\end{vmatrix}
\] | \[1 \times 4 - 2 \times 3 = 2 - 6 = -4\] |

Answer each question. Possible answers are given.

1. Complete so that each matrix has a positive determinant.

   \[ \begin{vmatrix}
   a & b \\
   c & d
   \end{vmatrix} \]

   \[ a \times b - b \times c > 0 \]

   a. \[ \begin{vmatrix}
   1 & 2 \\
   3 & 4
   \end{vmatrix} \]

   \[ 1 \times 4 - 2 \times 3 > 0 \]

   b. \[ \begin{vmatrix}
   -1 & 2 \\
   3 & 4
   \end{vmatrix} \]

   \[ -1 \times 4 - 2 \times 3 < 0 \]

2. Complete so that each matrix has a negative determinant.

   \[ \begin{vmatrix}
   a & b \\
   c & d
   \end{vmatrix} \]

   \[ a \times b - b \times c < 0 \]

   a. \[ \begin{vmatrix}
   -1 & 2 \\
   3 & 4
   \end{vmatrix} \]

   \[ -1 \times 4 - 2 \times 3 < 0 \]

   b. \[ \begin{vmatrix}
   1 & 2 \\
   -3 & 4
   \end{vmatrix} \]

   \[ 1 \times 4 - 2 \times (-3) > 0 \]

3. Complete so that each matrix has a determinant of 0.

   \[ \begin{vmatrix}
   a & b \\
   c & d
   \end{vmatrix} \]

   \[ a \times b - b \times c = 0 \]

   a. \[ \begin{vmatrix}
   1 & 2 \\
   -1 & -2
   \end{vmatrix} \]

   \[ 1 \times (-2) - 2 \times (-1) = 0 \]

   b. \[ \begin{vmatrix}
   1 & 2 \\
   -2 & -4
   \end{vmatrix} \]

   \[ 1 \times (-4) - 2 \times (-2) = 0 \]

4. Complete so that each matrix has a determinant of \(-1\).

   \[ \begin{vmatrix}
   a & b \\
   c & d
   \end{vmatrix} \]

   \[ a \times b - b \times c = -1 \]

   a. \[ \begin{vmatrix}
   1 & 2 \\
   -2 & -4
   \end{vmatrix} \]

   \[ 1 \times (-4) - 2 \times (-2) = -1 \]

   b. \[ \begin{vmatrix}
   1 & 2 \\
   -1 & -2
   \end{vmatrix} \]

   \[ 1 \times (-2) - 2 \times (-1) = -1 \]

5. Matrix \( W \) has a determinant of 0. What do you know about the dimensions and the entries of matrix \( W \)?

It must be a square matrix and the products of the diagonals are equal.